

## NEIGHBORHOOD CLASSIFICATION OF ISOTROPIC EMBEDDINGS

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### 1. The problem

If  $M$  is any manifold, and  $(P, \Omega)$  is a symplectic manifold, then an *isotropic* embedding of  $M$  in  $P$  is an embedding  $e: M \rightarrow P$  such that  $e^*\Omega = 0$ . (We refer the reader to [1], [3], or [6] for definitions and proofs omitted in this note.) A *neighborhood equivalence* from  $e_1: M_1 \rightarrow P_1$  to  $e_2: M_2 \rightarrow P_2$  consists of

- (i) a diffeomorphism  $g: M_1 \rightarrow M_2$ ,
- (ii) open neighborhoods  $U_i$  of  $e_i(M_i)$  in  $P_i$ ,
- (iii) a symplectomorphism  $f: U_1 \rightarrow U_2$  such that  $f \circ e_1 = e_2 \circ g$ .

We write  $f: e_1 \rightarrow e_2$ . The isotropic embeddings and neighborhood equivalences form a category  $\mathfrak{E}$ .

The *symplectic normal bundle*  $SN(e)$  of an isotropic embedding  $e: M \rightarrow P$  is a symplectic vector bundle over  $M$  whose fibre over  $m \in M$  is formed as follows. The image  $(Te)(T_m M)$  is an isotropic subspace of  $T_{e(m)}P$ ; the symplectic orthogonal space  $[(Te)(T_m M)]^\perp$  contains  $(Te)(T_m M)$ ; the quotient of the two, which is symplectic, is the fibre of  $SN(e)$ . Every neighborhood equivalence  $f: e_1 \rightarrow e_2$  induces a symplectic bundle isomorphism  $SN(f)$  from  $SN(e_1)$  to  $SN(e_2)$  covering a diffeomorphism from  $M_1$  to  $M_2$ ; we thus obtain a functor  $SN$  from  $\mathfrak{E}$  to the category  $\mathfrak{S}$  of symplectic vector bundles and bundle isomorphisms covering diffeomorphisms.

It is shown in [6] that the functor  $SN$  is surjective in the sense that every bundle isomorphism from  $SN(e_1)$  to  $SN(e_2)$  is  $SN(f)$  for some neighborhood equivalence  $f: e_1 \rightarrow e_2$ ; it is also shown that every symplectic vector bundle is isomorphic to  $SN(e)$  for some isotropic embedding  $e$ . Thus there is a one-to-one correspondence between neighborhood equivalence classes of isotropic embeddings and isomorphism classes of symplectic vector bundles.

The constructions in [6] leave something to be desired: the manifold into which  $M$  is embedded with a given symplectic normal bundle  $E$  is the

Whitney sum  $P = T^*M \oplus E$ , but the symplectic structure on  $P$  is not canonical, so bundle isomorphisms do not appear to lift to neighborhood equivalences. The purpose of this note is to improve the construction in [6] by finding a "symplectic thickening" functor  $St: \mathfrak{S} \rightarrow \mathfrak{E}$  which is a right inverse to  $SN$  in the sense that there is a natural transformation from  $ST \circ SN$  to the identity. To do so, we will use the construction in [7] of a phase space for a classical particle in a Yang-Mills field.

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## 2. The solution

Let  $E \rightarrow M$  be a symplectic vector bundle with fibre dimension  $2n$ . The frame bundle of  $E$  is the principal  $Sp(2n)$  bundle  $B \rightarrow M$  whose fibre over  $m$  is the manifold of linear symplectomorphisms from  $\mathbf{R}^{2n}$  to the fibre of  $E$  over  $m$ . The bundle associated to  $B \rightarrow M$  via the usual representation of  $Sp(2n)$  on  $\mathbf{R}^{2n}$  is just the original vector bundle  $E \rightarrow M$ .

The action of  $Sp(2n)$  on  $\mathbf{R}^{2n}$  preserves not only the symplectic structure  $\Omega = \sum_{i=1}^n dq_i \wedge dp_i$  but also the 1-form  $\omega = \frac{1}{2} \sum_{i=1}^n (p_i dq_i - q_i dp_i)$  for which  $d\omega = -\Omega$ . It follows that the action admits an equivariant momentum mapping  $\mu$  from  $\mathbf{R}^{2n}$  to the dual Lie algebra  $\mathfrak{sp}(2n)^*$ ; the mapping  $\mu$  is quadratic with  $\mu^{-1}(0) = \{0\}$ .

Given any principal  $G$ -bundle over a manifold  $M$ , and any symplectic  $G$ -manifold  $Q$  with an equivariant momentum mapping, the construction described in [7] produces a symplectic manifold  $P$  which can be fibred over  $T^*M$  with fibre  $Q$ . This fibration is associated to the pullback of the principal bundle from  $M$  to  $T^*M$ . The map  $P \rightarrow T^*M$  depends on the choice of a connection on the principal bundle, but the symplectic manifold  $P$  and the map  $P \rightarrow M$  do not.

Applying this construction with  $G = Sp(2n)$  and  $Q = \mathbf{R}^{2n}$ , we obtain a symplectic manifold  $P$  which can be fibred over  $T^*M$  with fibre  $\mathbf{R}^{2n}$ . This fibration is just the pullback of  $E$  to  $T^*M$ , which is the same thing as the Whitney sum  $T^*M \oplus E$ .

Now we must find a natural isotropic embedding from  $M$  to  $P$ . The idea is to construct a natural "zero section" from  $T^*M$  to  $P$ , even though the map  $P \rightarrow T^*M$  is not well-defined. To do so, we must look at the explicit construction of  $P$ .

According to [7], we must take the product symplectic manifold  $T^*B \times \mathbf{R}^{2n}$ , with its  $Sp(2n)$  action, and “reduce at  $0 \in \mathfrak{sp}(2n)^*$ , following the procedure of [4]. Specifically, we consider the momentum mapping  $\lambda: T^*B \rightarrow \mathfrak{sp}(2n)^*$  which is dual to the usual mappings  $\mathfrak{sp}(2n) \rightarrow T_b B$  onto the tangent spaces along the fibres of the principal bundle. Next, we take the submanifold  $\Sigma = \{(\beta, v) \in T^*B \times \mathbf{R}^{2n} | \lambda(\beta) = \mu(v)\}$ . Finally,  $P$  is the orbit space  $\Sigma/Sp(2n)$ .

To get a map  $P \rightarrow T^*M$ , we would need an  $Sp(2n)$ -equivariant projection from  $T^*B$  to  $T^*M$ , which is essentially a connection on  $B \rightarrow M$ . But let us restrict our attention to  $\lambda^{-1}(0)$ , which consists of those cotangent vectors to  $B$  which annihilate the fibres of  $B \rightarrow M$ . This set  $\lambda^{-1}(0)$  is naturally isomorphic to the pullback of  $T^*M$  to  $B$ . Now  $\Sigma$  contains as a submanifold  $\lambda^{-1}(0) \times \mu^{-1}(0) = \lambda^{-1}(0) \times \{0\}$ , which gives in  $P$  a submanifold  $[\lambda^{-1}(0) \times \{0\}]/Sp(2n) \approx \lambda^{-1}(0)/Sp(2n)$ , which may be identified with  $T^*M$  itself. Thus the zero section  $M \rightarrow T^*M$  gives an embedding  $e: M \rightarrow P$ .

Finally, one may check by using local trivializations of  $E$  that  $\lambda^{-1}(0)/Sp(2n)$  is a symplectic submanifold of  $P$  and that the tangent bundle to  $P$  along  $e(M)$  splits symplectically as the Whitney sum  $T^*M \oplus E$ . It follows that  $e$  is an isotropic embedding and that there is a natural isomorphism  $n(E)$  from  $SN(e)$  to  $E$ . Thus if we set  $ST(E) = e$ , we find that  $n$  is a natural transformation from  $SN \circ ST$  to the identity.

### 3. A remark

With the benefit of hindsight, we may see that the construction just described could have been “predicted” from Guillemin’s symbol calculus [2] for isotropic submanifolds of cotangent bundles. The quantization of  $P$  (see [3], [7], and the “dictionary” in [5]) consists of the sections of the bundle over  $M$  which is associated with the frame bundle of  $E$ , and whose typical fibre is a quantization of  $\mathbf{R}^{2n}$ . (We will ignore half-densities and half-forms in this remark.) A quantization of  $\mathbf{R}^{2n}$  is given by the space of rapidly decreasing smooth functions on  $\mathbf{R}^n$ , with the metaplectic representation. Thus, at least if  $E$  admits a metaplectic structure, the quantization of  $P$  is just a space of symplectic spinors as used in [2].

### References

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